Interactive Interference Alignment

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Abstract—We study interference channels (IFC) where interaction among sources and destinations is enabled, e.g., both sources and destinations can talk to each other. The interaction can come in two ways: 1) for half-duplex radios, destinations can talk back to sources using either simultaneous out-of-band (white spaces) transmission or in-band half-duplex transmission; 2) for full-duplex radios, both sources and destinations can transmit and listen in the same channel simultaneously. The flexibility afforded by interaction among sources and destinations allows for the derivation of interference alignment (IA) strategies that have desirable "engineering properties": insensitivity to the rationality or irrationality of channel parameters, small block lengths and finite SNR operations. We show that for several classes of interference channels the interactive interference alignment scheme can achieve the optimal degrees of freedom. We also give simulation results on the finite SNR performance of interactive interactive alignment for 3-user and 4-user half-duplex interference channels.

I. INTRODUCTION

Due to the broadcast nature of wireless communication, an active transmitter will create interference to all other receivers using the same channel (in time and frequency). This is a basic challenge in supporting concurrent reliable wireless communications. Traditional methods have involved power control, scheduling and antenna beamforming/nulling to manage interference. Interference alignment is a novel interference management technique which makes concurrent wireless communications feasible [1], [2].

Since its introduction, substantial research has been conducted in understanding the gains of interference alignment. From a theoretical perspective, the focus has been mainly in understanding the degrees of freedom (first term in the high SNR approximation of the information theoretic capacity region). A seminal result, shown in [1], is that we can communicate at a total $\frac{K}{2}$ degrees of freedom for nearly all time-varying/frequency selective K-user interference channel (this is also an upper bound). To get to $\frac{K}{2}$ degrees of freedom, the scheme employed (vector space interference alignment) requires the channel diversity to be unbounded; in fact, one needs the channel diversity to grow like K^{2K^2} [3]. The diversity order required is huge even for modest number of users K (such as 3 and 4). This is definitely so in a practical communication system, where there is hardly enough channel diversity to make such a vector space interference alignment scheme feasible.

A practically relevant problem is to understand the fundamental degrees of freedom for a fixed deterministic channel. For fully connected channel matrix H, the total degrees of freedom are upper bounded by $\frac{K}{2}$ and [4] shows that this upper bound is achievable for almost all channel matrices H using a coding scheme based on Diophantine approximation. However, this result is limited in two ways. First, the coding scheme is very sensitive to whether entries of H are rational or irrational. Second, although it is provable $\frac{K}{2}$ degrees of freedom are achievable for almost all H, for a given H, in general it is not known what are the optimal achievable degrees of freedom. [5] shows that the $\frac{K}{2}$ result for almost all H can be derived using Rényi information dimension. Again, the result is sensitive to whether entries of H are rational or irrational, and for fixed channel matrix H, in general the optimal achievable degrees of freedom is unknown. The recent works of [6], [7] address this issue to a good extent for the case of the two-user X channel and the symmetric K-user interference channel respectively, but the engineering implication of the proposed coding schemes remains unclear.

Therefore, despite significant theoretical progress on the K-user interference channel problem, it is still unclear how to make interference alignment practical. The drawbacks of existing schemes may be inherent to the channel model which assumes sources can only transmit and destinations can only listen, while in practice radios can both transmit and receive. We study new channel models where interaction among sources and destinations is enabled, e.g., both source and destination can talk to each other. The interaction can come in two ways: 1) for half-duplex radios, destinations can talk back to sources using simultaneous out-of-band transmission or in-band half-duplex transmission; 2) for full-duplex radios, both sources and destinations can transmit and listen in the same channel simultaneously [8]. Although [9] shows that for interference channel, relays, feedback, and full-duplex operation cannot improve the degrees of freedom beyond $\frac{K}{2}$, we demonstrate that the interaction among sources and destinations enables flexibility in designing simple interference alignment scheme and in several cases achieves the optimal degrees of freedom.

Our main contribution is to propose a simple interference alignment scheme by exploiting the interactions among sources and destinations, and prove that the scheme can achieve the optimal degrees of freedom for several classes of interference channels, including half-duplex 3-user IFC, full-duplex 3-user and 4-user IFC, and full-duplex 4-user MIMO IFC. One specific aspect of our model, namely, feedback using the reciprocal interference channel, has been considered in prior work [10]. In this work, we improve on this state-of-the-

art in two ways: we prove new results for this specific model and also generalize this model to exploit more general modes of interaction in order to achieve interference alignment. In addition, we do extensive numeric simulations and show the proposed interactive interactive alignment scheme also works for some other classes of IFC empirically. We use tools from algebraic geometry to show why success of numeric simulations can suggest that the scheme should work well for almost all channel parameters in a rigorous way.

The paper is organized as follows. We describe the channel model in Section II and give some motivating examples in Section III. We present the interactive communication scheme for half-duplex radios and interference alignment conditions in Section IV, and our main technical results from algebraic geometry on the interference alignment feasibility in Section V. Section VI and Section VII study interactive interference alignment for half-duplex and full-duplex K-user interference channels with small K, respectively. Simulation results on the finite SNR performance evaluation are presented in Section VIII. Section IX discusses a general multi-phase interactive communication scheme for K-user interference channel with large K, and lists several open problems. Section X concludes this paper.

II. SYSTEM MODEL

Consider a K-user interference channel with $2K$ radios, where radios $s_1, s_2, \ldots, s_K$ are sources and radios $t_1, t_2, \ldots, t_K$ are destinations. For each $1 \leq i \leq K$, source s_i wants to send an independent message W_i to destination t_i.

Let $H \in \mathbb{C}^{K \times K}$ denote the forward channel matrix from sources to destinations, and the input and output signals of the forward channel are related as

$$y_i[n] = \sum_{j=1}^{K} H_{ij} x_j[n] + z_i[n], \tag{1}$$

where n is the time index, $y_i[n]$ is the signal received by destination t_i at time n, $x_j[n]$ is the signal sent out by source s_j at time n, $z_i[n]$ is the channel noise, and H_{ij} is the channel coefficient from source s_j to destination t_i.

The above is a canonical channel model for K-user interference channel.

In this work, we consider two channel models where interaction among sources and destinations can be enabled.

A. Interaction from destinations to sources via reverse transmission for half-duplex radios

Since in practice radios can both actively talk (send signals) or passively listen (receive signals), we assume destinations can also actively talk back to sources either via simultaneous out-of-band (white spaces) transmission or via in-band half-duplex transmission. Let $G \in \mathbb{C}^{K \times K}$ denote the feedback channel (or reverse channel) matrix from destinations to sources. Similarly, the input-output relation of the feedback
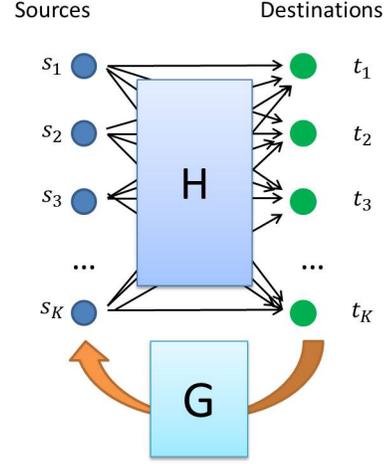

Fig. 1. System Model For Half-duplex Radios: $s_1, s_2, \ldots, s_K$ are sources and $t_1, t_2, \ldots, t_K$ are destinations. H is the forward channel matrix from sources to destinations, and G is the feedback channel matrix from destinations to sources.

channel is

$$f_i[n] = \sum_{j=1}^{K} G_{ij} v_j[n] + \tilde{z}_i[n], \tag{2}$$

where n is the time index, $v_j[n]$ is the signal sent out by destination t_j at time n, $f_i[n]$ is the signal received by source s_i at time n, $\tilde{z}_i[n]$ is the channel noise, and G_{ij} is channel coefficient from destination t_j to source s_i.

If the destinations use in-band half-duplex transmissions, due to the reciprocity of wireless channels, we have

$$G = H^T, \tag{3}$$

where H^T denotes the transpose of H.

B. Interaction via transmitting and receiving simultaneously for full-duplex radios

Full-duplex radios can both send and receive signals using the same channel simultaneously, and this capability naturally enables the interaction among all radios in the network.

We assume that all sources and destinations have full-duplex antennas, and all nodes can transmit and receive signals using the same channel simultaneously. Let $H \in \mathbb{C}^{K \times K}$ be the channel matrix from sources to destinations, $U \in \mathbb{C}^{K \times K}$ be the channel matrix among sources, and $W \in \mathbb{C}^{K \times K}$ be the channel matrix among destinations. The input-output relation of this full-duplex communication channel is

$$y_i[n] = \sum_{j=1}^{K} H_{ij} x_j[n] + \sum_{j=1}^{K} W_{ij} v_j[n] + z_i[n], \tag{4}$$

$$f_i[n] = \sum_{j=1}^{K} H_{ji} v_j[n] + \sum_{j=1}^{K} U_{ij} x_j[n] + \tilde{z}_i[n], \tag{5}$$

where n is the time index, $x_j[n]$ and $v_j[n]$ are the signals sent out by source s_j and destination t_j at time n, respectively,
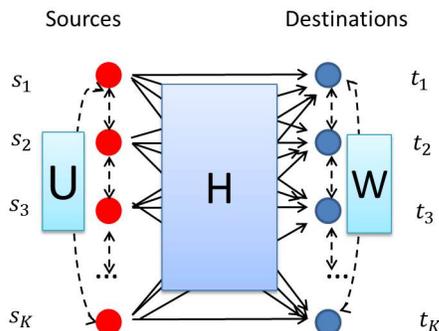
Fig. 2. System Model For Full-duplex Radios: $s_1, s_2, \ldots, s_K$ are sources and $t_1, t_2, \ldots, t_K$ are destinations. H is the forward channel matrix from sources to destinations, U is the channel matrix among sources, and W is the channel matrix among destinations.

$y_i[n]$ and $f_i[n]$ are the signal received by destination t_i and source s_i at time n, $z_i[n]$ and $\tilde{z}_i[n]$ are the channel noise, H_{ij} is channel coefficient from s_j to t_i and also the channel coefficient from t_i to s_j due to channel reciprocity, W_{ij} is the channel coefficient from r_j to r_i, and U_{ij} is the channel coefficient from s_j to s_i.

III. MOTIVATING EXAMPLES

The main theme of this paper is to show how interaction among destinations and sources can help do interference alignment/neutralization and thus make concurrent wireless communication feasible.

In this section, we start with a simple example of 3-user half duplex interference channel where communication is impossible without interaction from destinations, and describe a simple three-phase transmission scheme which makes concurrent communication feasible by exploiting the interaction from destinations to sources. Then, we generalize this three-phase transmission scheme to a class of K-user interference channels where the forward channel matrix H can be written as the sum of a diagonal matrix and a rank 1 matrix.

A. A symmetric 3-user interference channel

Consider a 3-user interference channel with forward channel matrix

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{6}$$

where all interference link gains are 1, and all direct link gains are 0.

Since the direct links have gains 0, it is trivial to see that without interaction from destinations, no reliable communication is possible. However, if we allow destinations to talk back to sources using the reciprocal channel, we can achieve $\frac{3}{2}$ degrees of freedom as seen below.

Consider the following three-phase interactive transmission scheme:

- Phase 1: All sources send their independent symbols x_1, x_2, x_3 simultaneously. Then destinations receive

$$y_1 = x_2 + x_3 + \text{noise}, \tag{7}$$
$$y_2 = x_1 + x_3 + \text{noise}, \tag{8}$$
$$y_3 = x_1 + x_2 + \text{noise}. \tag{9}$$

- Phase 2: After receiving signal y_1, y_2, y_3 from sources in Phase 1, all destinations simultaneously send out $-y_i$ to sources using the reciprocal channel, and sources get

$$\begin{aligned} f_1 &= -y_2 - y_3 + \text{noise} \\ &= -2x_1 - x_2 - x_3 + \text{noise}, \end{aligned} \tag{10}$$
$$\begin{aligned} f_2 &= -y_1 - y_3 + \text{noise} \\ &= -2x_2 - x_1 - x_3 + \text{noise}, \end{aligned} \tag{11}$$
$$\begin{aligned} f_3 &= -y_1 - y_2 + \text{noise} \\ &= -2x_3 - x_1 - x_2 + \text{noise}. \end{aligned} \tag{12}$$

- Phase 3: Now each source s_i has two sets of signals x_i and f_i, and each s_i can simultaneously send out $(-3x_i - f_i)$ to destinations using the forward channel. destinations will get

$$\begin{aligned} y'_1 &= (-3x_2 - f_2) + (-3x_3 - f_3) + \text{noise} \\ &= 2x_1 + \text{noise}, \end{aligned} \tag{13}$$
$$\begin{aligned} y'_2 &= (-3x_1 - f_1) + (-3x_3 - f_3) + \text{noise} \\ &= 2x_2 + \text{noise}, \end{aligned} \tag{14}$$
$$\begin{aligned} y'_3 &= (-3x_1 - f_1) + (-3x_2 - f_2) + \text{noise} \\ &= 2x_3 + \text{noise}. \end{aligned} \tag{15}$$

Therefore, after two forward transmissions and one reverse transmission, each destination t_i will get the desired symbol x_i from the corresponding source s_i without any interference. If we do not count the reverse transmission, the total degrees of freedom achieved are $\frac{3}{2}$.

B. A symmetric K-user interference channel with special channel matrix

We can extend the above result to K-user interference channel where the channel matrix is an all-one matrix except all diagonal entries being zero for any $K \geq 3$.

Consider a K-user interference channel where the channel matrix H satisfies

$$H_{ij} = 1, \forall i \neq j \tag{16}$$
$$H_{ii} = 0, \forall 1 \leq i \leq K. \tag{17}$$

We can show that the following three-phase interactive transmission scheme with two forward transmissions and one reverse transmission can make each destination t_i get the desired symbol from source s_i for all $1 \leq i \leq K$.

- Phase 1: All sources send their independent symbols $x_1, x_2, \ldots, x_K$ simultaneously. Then each destination t_i get

$$y_i = \sum_{1 \leq j \leq K, j \neq i} x_j + \text{noise}. \tag{18}$$

- Phase 2: After receiving signal $y_1, y_2, \ldots, y_K$ from sources in Phase 1, all destinations simultaneously send out $-y_i$ to sources using the reciprocal channel. Then each source s_i gets

$$
\begin{aligned}
f_i &= \sum_{1 \leq j \leq K, j \neq i} y_j + \text{noise} \\
&= -(K-1)x_i - \sum_{1 \leq j \leq K, j \neq i} (K-2)x_j + \text{noise}.
\end{aligned}
\tag{19}
$$

- Phase 3: Now each source s_i has two sets of signals x_i and f_i, and each s_i can simultaneously send out

$$
(-(K^2 - 3K + 3)x_i - f_i) \tag{20}
$$

to destinations using the forward channel. Then each destination t_i will get

$$
\begin{aligned}
y_i' &= \sum_{1 \leq j \leq K, j \neq i} (-(K^2 - 3K + 3)x_i - f_i) + \text{noise} \\
&= (K-2)(K-1)x_i + \text{noise}.
\end{aligned}
\tag{21}
$$

For $K \geq 3$, $(K-2)(K-1) \neq 0$. Therefore, each destination t_i will get the desired symbol from the source s_i without any interference for all $1 \leq i \leq K$.

In fact, we can show that if $H \in \mathbb{C}^{K \times K}$ can be written as the sum of diagonal matrix and a rank 1 matrix, then in general two forward transmissions and one reverse transmission can make each destination t_i get the desired symbol from source s_i for all $1 \leq i \leq K$.

Theorem 1. *Consider a special K-user interference channel with reciprocal feedback channel ($G = H^T$), where the forward channel matrix H can be written as the sum of a diagonal matrix and a rank 1 matrix, i.e.,*

$$
H = D + uv^T, \tag{22}
$$

where $D \in \mathbb{C}^{K \times K}$ is a diagonal matrix and $u, v \in \mathbb{C}^{K \times 1}$ are column vectors.

If D is invertible, each component of u and v is nonzero, and

$$
\alpha \triangleq 3 + 3v^T D^{-1} u + (v^T D^{-1} u)^2 \neq 1, \tag{23}
$$

then there exists a three-phase interactive transmission scheme such that two forward transmissions and one reverse transmission can make each destination t_i get the desired symbol from source s_i without any interference for all $1 \leq i \leq K$.

Proof: See Appendix A. ∎

IV. COMMUNICATION SCHEME AND INTERFERENCE ALIGNMENT CONDITIONS

The three-phase interactive transmission scheme described in Section III is a natural scheme to exploit the interaction from destinations to sources to make concurrent wireless communication feasible. We can generalize this transmission scheme in the following:
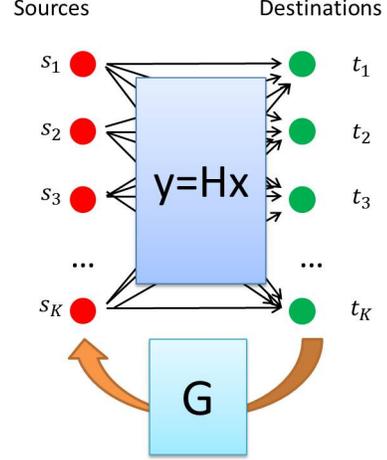

Fig. 3. Phase 1: All sources send their independent symbols simultaneously. And destinations get $y = Hx + $ noise.

- Phase 1 (forward transmission): All sources send their independent symbols simultaneously. And destinations get $y = Hx + n$, where n is the additive noise of the channel.
- Phase 2 (interaction from destinations): After receiving signals from sources in phase1, all destinations scale y and send back to sources using the reverse channel. sources get $f = GD_1 y + \tilde{n}$, where $\tilde{n}$ is the additive noise of the channel. Since each source and each destination only knows what signals they sent out and received, the coding matrix D_1 has to be diagonal.
- Phase 3 (forward transmission): Now each source has two sets of signals x and f, and sources can send out a linear combination of x and f to destinations via the forward channel. destinations get

$$
\begin{aligned}
y' &= H(D_2 x + D_3 f) + \hat{n} \tag{24} \\
&= (HD_2 + HD_3 GD_1 H)x + HD_3 GD_1 n + HD_3 \tilde{n} + \hat{n}. \tag{25}
\end{aligned}
$$

Again, the matrices D_2, D_3 are constrained to be diagonal.

In the above scheme, since each source and each destination only know what signals they sent out and received, the coding matrices D_1, D_2 and D_3 have to be diagonal, i.e., each node can only do coding over what signals it sent out and received.

To make destinations be able to decode desired signals, one possibility is that we choose diagonal matrices D_1, D_2 and D_3 such that $(HD_2 + HD_3 GD_1 H)$ is a diagonal matrix and every diagonal entry is nonzero, which makes each destination get the desired signal from y' without any interference. In fact we have used coding matrices with this property for the special class of interference channels presented in Section III. So a natural question to ask is whether there always exists such D_1, D_2 and D_3.

Requiring that all off-diagonal entries be zero gives rise to $K(K-1)$ polynomial equations, and each diagonal entry
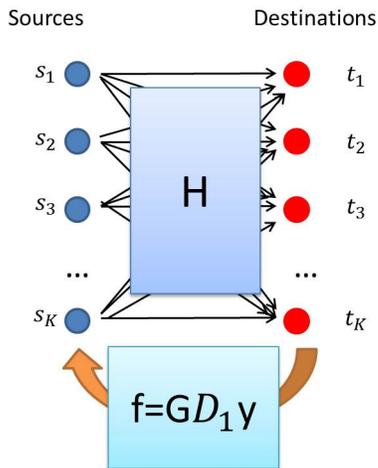
Fig. 4. Phase 2: After receiving signals from sources in phase1, all destinations scale y and send back to sources using the feedback channel. Sources get $f = GD_1y + \text{noise}$.
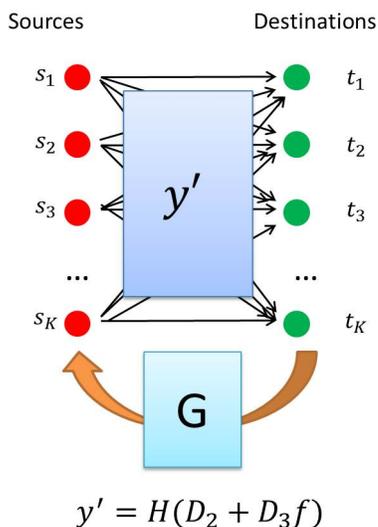

$$y' = H(D_2 + D_3f)$$

Fig. 5. Phase 3: Now each source has two sets of signals x and f, and sources can send out a linear combination of x and f to destinations via the forward channel. Destinations get $y' = H(D_2x + D_3f) + \text{noise}$.

being nonzero leads to K inequalities, whereas in total there are only $3K$ variables, corresponding to the K variables in each of the diagonal coding matrices D_1, D_2 and D_3. One may therefore be led to conjecture that if the number of variables is more than the number of equations, a solution should exist that simultaneously satisfies all equations and inqualities. Unfortunately, this conjecture is not true: there are cases where even though the number of variables is more than equations, due to the coupled constraints on the equations and inequalities, all off-diagonal entries of $(HD_2 + HD_3GD_1H)$ being zero implies that at least one diagonal entry is zero. A concrete example is the 3-user interference channel with reciprocal reverse channel, which will be discussed in Section VI.

One way to overcome this problem is to let the receivers decode the signal based on both y (which the destinations received during the first phase) and y' instead of relying purely on y' alone (as done in the above strategy). Now, the destinations can cancel out all interference in y' by taking a linear combination of y and y' if the interference terms in y' are aligned with the interference terms in y.

More precisely, let $B \triangleq HD_2 + HD_3GD_1H$. Then for destination t_i, in Phase 1 it receives

$$y_i = H_{ii}x_i + \sum_{j \neq i} H_{ij}x_j + \text{noise}, \qquad (26)$$

and in Phase 3 it receives

$$y'_i = B_{ii}x_i + \sum_{j \neq i} B_{ij}x_j + \text{noise}. \qquad (27)$$

If for all $1 \leq i \leq K$, we have

$$\frac{B_{ij}}{H_{ij}} = \frac{B_{ij'}}{H_{ij'}}, \quad \forall j, j' \neq i, \qquad (28)$$

or equivalently

$$\frac{B_{ij}}{H_{ij}} = \lambda_i, \quad \forall j \neq i, \qquad (29)$$

for some constant λ_i, then interference terms of y and y' for each destination are aligned with each other. So destination t_i can compute $y_i - \lambda_i y'_i$ and get

$$(H_{ii} - \lambda_i B_{ii})x_i + \text{noise}. \qquad (30)$$

Therefore, if $(H_{ii} - \lambda_i B_{ii}) \neq 0$, or equivalently,

$$\frac{B_{ii}}{H_{ii}} \neq \frac{B_{ij}}{H_{ij}}, \quad \forall j \neq i, \qquad (31)$$

then destination t_i can get the desired signal sent by source s_i without any interference.

Without introducing the auxiliary variables λ_i, aligning $(K - 1)$ interference terms for each destination corresponds to $(K - 2)$ equations, and thus in total we have $K(K-2)$ interference alignment equations. Preserving the desired signals after canceling all interference terms leads to K inequalities. Therefore, to make the three-phase communication scheme work and thus achieve the optimal $\frac{K}{2}$ degrees of freedom, we need to solve $K(K - 2)$ equations with $3K$ variables and check that whether the solution satisfies the K inequalities.

In some cases, we can reduce the polynomial equations corresponding to interference alignment equality constraints to linear equations, the solution of which has a closed-form expression and thus it can be easily verified whether the inequality constraints are satisfied. However, in general the system of polynomial equations is nonlinear, and it may not have closed-form solution, making it hard to check whether the inequality constraint can be satisfied. Our main approach is to convert the system of polynomial equations and polynomial inequalities to a system of polynomial equations, and then use tools from algebraic geometry to check the existence of solution for the system of polynomial equations.

First, we show how to reduce the problem to checking the existence of solutions to a system of polynomial equations. Then we will present our main technique for solving the polynomial system using tools from algebraic geometry in Section V.

Suppose there are N polynomial equation constraints $f_i = 0$, $1 \leq i \leq N$, and M polynomial inequality constraints $g_j \neq 0$, $1 \leq j \leq M$, over S variables $\{d_1, d_2, \ldots, d_S\}$. By introducing an auxiliary variable t, we define a polynomial function $\hat{g}$ as

$$\hat{g} \triangleq t \prod_{j=1}^{M} g_j - 1. \tag{32}$$

Lemma 1. *There exists $(d_1, d_2, \ldots, d_S)$ satisfying equality and inequality constraints*

$$f_i = 0, \quad \forall 1 \leq i \leq N \tag{33}$$
$$g_j \neq 0, \quad \forall 1 \leq j \leq M, \tag{34}$$

if and only if there exists $(d_1, d_2, \ldots, d_S, t)$ satisfying

$$f_i = 0, \quad \forall 1 \leq i \leq N \tag{35}$$
$$\hat{g} = 0. \tag{36}$$

Proof: If $(d_1, d_2, \ldots, d_S)$ is a solution to (33) and (34), then $(d_1, d_2, \ldots, d_S, t)$ is a solution to (35) and (36), where

$$t \triangleq \frac{1}{\prod_{j=1}^{M} g_j}. \tag{37}$$

On the other hand, if $(d_1, d_2, \ldots, d_S, t)$ is a solution to (35) and (36), then $(d_1, d_2, \ldots, d_S)$ satisfies both (33) and (34). ■

Therefore, due to Lemma 1, the problem of checking the existence of solutions to a system of polynomial equations and inequalities can be reduced to the problem of checking existence of solutions of a system of polynomial equations, which is well studied for algebraically closed field in algebraic geometry [11].

V. GENERAL SOLUTION METHODOLOGY

In this section we present our main technical results on checking the existence of solutions to the interference alignment equations for generic channel parameters H and G using tools in algebraic geometry. For more details on algebraic geometry, we refer the reader to Appendix B and the excellent textbook [11].

We follow the standard notation in [11]. Let k denote a field, and let $k[\xi_1, \xi_2, \ldots, \xi_n]$ denote the set of all polynomials in the variables $\xi_1, \xi_2, \ldots, \xi_n$ with coefficients in k.

Definition 1 (Definition 1 on page 5 of [11])**.** *Let k be a field, and let $f_1, \ldots, f_s$ be polynomials in $k[\xi_1, \xi_2, \ldots, \xi_n]$. Then we call*

$$V(f_1, \ldots, f_s) \triangleq \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0,$$
$$\forall 1 \leq i \leq s\} \tag{38}$$

*the **affine variety** defined by $f_1, \ldots, f_s$.*

As discussed in Section IV, our problem can be reduced to checking the existence of solutions to a system of polynomial equations. In the language of algebraic geometry, the problem is to check whether the affine variety defined by some polynomials is an empty set or not, which is well studied for algebraically closed field in algebraic geometry. In wireless communication, the channel coefficients are represented as complex numbers $\mathbb{C}$, which is an algebraically closed field.

The standard approach to checking whether an affine variety is an empty set is to use Buchberger's algorithm to compute the Gröbner basis of the given polynomials, and from the Gröbner basis we can easily conclude whether the corresponding affine variety is empty or not [12].

One important implication of these results in algebraic geometry is that if the coefficients in the polynomial equations are rational functions of variables $\{h_i\}$, then except a set of $\{h_i\}$ which satisfies a nontrivial polynomial equation on $\{h_i\}$ and thus has a measure 0, either for all numeric values of $\{h_i\}$ the polynomial equations have a solution, or for all numeric values of $\{h_i\}$ the polynomial equations do not have a solution.

More precisely,

Theorem 2. *Let $f_1, \ldots, f_s$ be polynomials in $k[\xi_1, \xi_2, \ldots, \xi_n]$, where $k = \mathbb{C}$ and all coefficients of the polynomials are rational functions of variables $h_1, h_2, \ldots, h_m$. Then there exists a nontrivial polynomial equation on $h_1, h_2, \ldots, h_m$, denoted by $e(h_1, h_2, \ldots, h_m)$, such that except the set of $\{(h_1, h_2, \ldots, h_m) \mid e(h_1, h_2, \ldots, h_m) = 0\}$, either for all $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1 \times m}$, $V(f_1, \ldots, f_s) \neq \emptyset$, or for all $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1 \times m}$, $V(f_1, \ldots, f_s) = \emptyset$.*

Proof: See Appendix C. ■

For the polynomial equations describing the interference alignment problem, the coefficients of the polynomials are rational functions of channel parameters H and G in symbolic form. Therefore, in the context of the interference alignment feasibility problem, this main result can be restated as follows. *Either one* of the following two statements hold:

- For almost all[1] channel realizations of H and G, there exists solution to the system of interference alignment equations.
- Or, for almost all channel realizations of H and G, there does not exist solution to the system of interference alignment equations.

Although in theory we can compute in finite number of steps the symbolic Gröbner basis for these polynomials with symbolic coefficients to check whether for almost all H and G there exists solutions, it turns out to be computationally infeasible to run the Buchberger's algorithm for the symbolic polynomial equations for most of our interference alignment problems, due to the fact that the orders of intermediate symbolic coefficients can increase exponentially. However, it

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[1]We emphasize that in our results "almost all" means for all numeric values except a set of parameters which satisfy a nontrivial polynomial equation. Therefore, in contrast to results in [5] and [4], our results are not sensitive to whether the channel parameters are rational or irrational.

is much easier to compute a Gröbner basis numerically. Due to Theorem 2,

Corollary 1. *If we draw the channel parameters according to a continuous probability distribution, then with probability one the numeric polynomial equations have a solution if and only if for almost all channel realizations the polynomials equations have solution.*

Hence, while we may not be able to prove that certain polynomial equations have solution for almost all channel parameters because of computational difficulty, numeric simulations can let us make claims with high credibility.

VI. INTERACTIVE INTERFERENCE ALIGNMENT FOR SMALL K WITH HALF DUPLEX RADIOS

In this section, we study interactive interference alignment for half-duplex K-user interference channel with small K.

A. $K = 3$

As discussed in Section IV, to make the three-phase interactive communication scheme work, ideally we would like to find diagonal coding matrices D_1, D_2 and D_3 to make $(HD_2 + HD_3GD_1H)$ be a diagonal matrix with each diagonal entry being nonzero, so that interference neutralization can be achieved in the third phase. For 3-user interference channel, the number of interference neutralization equations is 6, and the number of variables is $3K = 9$. While it seems we can solve the interference neutralization equations with more variables, due to the coupled constraints on the equations and inequalities, all off-diagonal entries of $(HD_2 + HD_3GD_1H)$ being zero implies that at least one diagonal entry is zero. A simple counter example is the case when the reverse channel is reciprocal to the forward channel, i.e., $G = H^T$. We did extensive numeric simulations, and in each numeric instance we find that there is no solution to solve both equations and inequalities by using the methodology (computing Gröbner basis) introduced in section V. Therefore, due to Corollary 1, we believe the following claim holds:

Claim 1. *For 3-user interference channel with reciprocal feedback channel ($G = H^T$), for generic matrix[2] H, there does not exist D_1, D_2 and D_3 such that $(HD_2 + HD_3GD_1H)$ is a diagonal matrix and every diagonal entry is nonzero.*

As discussed in Section IV, it is not necessary to do exact interference neutralization. Instead, interference alignment is also sufficient. When $K = 3$, the number of variables is $3K = 9$, which is much more than the number of interference alignment equations $K(K-2) = 3$, so it is not surprising that we can find D_1, D_2 and D_3 to solve the interference alignment equations and inequalities. [10] gives an analytical solution to do interference alignment for 3-user interference channel with feedback. Here, we present a different proof technique, which also will be applied in interference alignment with full-duplex antennas in Section VII.

First we state a lemma which will be used in the proof of Theorem 3.

Lemma 2. *Let $S_0 \subset \mathbb{C}^n$ be a linear subspace with dimension d_0. For any m linear subspaces $S_1, S_2, \ldots, S_m \subset S_0$ with dimensions $d_1, \ldots, d_m$ respectively, if*

$$d_i < d_0, \quad \forall 1 \leq i \leq m, \tag{39}$$

then $S_0 \backslash (\cup_{i=1}^{m} S_i)$ is a nonempty set.

Proof: Let $e_1, \ldots, e_{d_0} \in \mathbb{C}^n$ be a set of bases of S_0. Let $A \in \mathbb{C}^{d_0 \times d_0 m}$ such that every d_0 by d_0 submatrix of A has full rank. It is easy to show the existence of A with such property. Indeed, if every entry of A is independently generated according to Gaussian distribution (or other continuous probability distributions), then with probability 1 every d_0 by d_0 submatrix of A has full rank.

Define $d_0 m$ vectors $b_1, b_2, \ldots, b_{d_0 m}$ by setting

$$b_i \triangleq \sum_{j=1}^{d_0} A_{ji} e_j, \tag{40}$$

for all $1 \leq i \leq d_0 m$. Since every d_0 by d_0 square submatrix of A has full rank, every d_0 vectors of $b_1, b_2, \ldots, b_{d_0 m}$ are linearly independent.

Suppose $S_0 \backslash (\cup_{i=1}^{m} S_i)$ is a nonempty set, i.e., $S_0 = \cup_{i=1}^{m} S_i$. Then $b_1, b_2, \ldots, b_{d_0 m} \in \cup_{i=1}^{m} S_i$. By pigeonhole principle, there exists $i^* \leq d_0 m$ such that S_{i^*} contains at least d_0 vectors of $b_1, b_2, \ldots, b_{d_0 m}$. Since every d_0 vectors of $b_1, b_2, \ldots, b_{d_0 m}$ are linearly independent, the dimension of S_{i^*} is no less than d_0, contradicting with the fact that $d_{i^*} < d_0$.

Therefore, $S_0 \backslash (\cup_{i=1}^{m} S_i)$ is a nonempty set. ∎

Theorem 3. *For 3-user interference channel with reciprocal feedback channel ($G = H^T$), and for generic channel matrix H, there exists solutions to the interference alignment equations and inequalities.*

Proof: The $K(K-2) = 3$ equations are

$$\frac{B_{12}}{H_{12}} = \frac{B_{13}}{H_{13}}, \tag{41}$$

$$\frac{B_{21}}{H_{21}} = \frac{B_{23}}{H_{23}}, \tag{42}$$

$$\frac{B_{31}}{H_{31}} = \frac{B_{32}}{H_{32}}, \tag{43}$$

and the $K = 3$ inequalities are

$$\frac{B_{11}}{H_{11}} \neq \frac{B_{12}}{H_{12}}, \tag{44}$$

$$\frac{B_{22}}{H_{22}} \neq \frac{B_{23}}{H_{23}}, \tag{45}$$

$$\frac{B_{33}}{H_{33}} \neq \frac{B_{32}}{H_{32}}, \tag{46}$$

Since the number of variables is much more than the number of equations, we do not need to use all the variables. In fact,

we can set D_1 to be a simple diagonal matrix

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (47)$$

Write D_2 and D_3 as

$$D_2 = \begin{pmatrix} D_{21} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{23} \end{pmatrix}, \qquad (48)$$

$$D_3 = \begin{pmatrix} D_{31} & 0 & 0 \\ 0 & D_{32} & 0 \\ 0 & 0 & D_{33} \end{pmatrix}. \qquad (49)$$

Then equations (41), (42) and (43) are homogeneous linear equations in terms of variables $D_{21}, D_{22}, D_{23}, D_{31}, D_{32}$ and D_{33}. Let S denote the set of solutions to linear equations (41), (42) and (43). It is easy to check that the corresponding coefficient matrix has a rank of 3. Therefore, S is a linear subspace of dimension $6 - 3 = 3$.

Consider the following equations corresponding to the inequality constraints

$$\frac{B_{11}}{H_{11}} = \frac{B_{12}}{H_{12}}, \qquad (50)$$

$$\frac{B_{22}}{H_{22}} = \frac{B_{23}}{H_{23}}, \qquad (51)$$

$$\frac{B_{33}}{H_{33}} = \frac{B_{32}}{H_{32}},. \qquad (52)$$

Let S_1 denote the set of solutions to (41), (42), (43) and (50), let S_2 denote the set of solutions to (41), (42), (43) and (51), and let S_3 denote the set of solutions to (41), (42), (43) and (52). We can check that all the coefficient matrices of the linear equations have a rank of 4, so all of S_1, S_2 and S_3 are linear subspaces of dimension $6 - 4 = 2$.

Now consider the set $S^* \triangleq S \backslash (S_1 \cup S_2 \cup S_3)$. Any element in S^* satisfies both the equations (41), (42), (43), and the inequality constraints (44), (45), (46).

Since S is linear subspace of dimension 3, which is larger than the dimensions of S_1, S_2 and S_3, S^* is a nonempty set by Lemma 2. This completes the proof. ∎

The same result continues to hold when the feedback channel matrix is also a generic matrix independent of H.

It is worth pointing out that the 3-user interference channel under an alternate local feedback model, where each receiver sends feedback to its corresponding transmitter, was studied in [13] recently. In contrast to our model, where obtaining feedback using natural reciprocal channel takes up a single time slot, local feedback can take 3 time slots to implement.

B. $K = 4$

For 4-user interference channel, we have $K(K - 2) = 8$ polynomial equation constraints and $K = 4$ inequality constraints with $3K = 12$ variables. We cannot use the same technique as in the case of three-user interference channel by setting D_1 or D_3 to be a deterministic constant matrix and

thus reducing the polynomial equations to linear equations. The reason is that if we do not use D_1 or D_3 as variables, the number of equations will be the same as the number of variables, and this will prevent us from finding a solution which also satisfies the inequality constraints. Instead, we use the tools introduced in Section V to study the feasibility of interactive interference alignment. More precisely, we first convert the system of polynomial equations and polynomial inequalities to a system of polynomial equations, and then we check whether it has solutions or note by computing the Gröbner basis of corresponding polynomials.

Due to computational difficulty, it is hard to compute the Gröbner basis for the symbolic polynomials. We do extensive numeric verifications: each time we generate random numeric channel matrices, and then apply Buchberger's algorithm to compute a Gröbner basis. In all numeric instances, the Gröbner basis does not contain a nonzero scalar, which implies that solution to (28) and (31) exists. Therefore, due to Corollary 1, although we cannot compute a symbolic Gröbner basis explicitly, from numeric verifications we believe the following claims hold.

Claim 2. *For 4-user interference channel with reciprocal feedback channel ($G = H^T$), for generic matrix H, there exists D_1, D_2 and D_3 such that each row of $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries.*

Claim 3. *For 4-user interference channel with out-of-band feedback channel, for generic matrices H and G, there exists D_1, D_2 and D_3 such that each row of $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries.*

[10] formulates the interference alignment problem as a rank constrained nonconvex optimization problem, and reports that all numeric simulations confirm the existence of solutions to the interference alignment equations and inequalities.

In some cases when H has special structures, we are able to compute the symbolic Gröbner basis of the interference alignment polynomials, and thus we can conclude whether for almost all H with such structure there exists interference alignment solutions. One positive example is that if H is a symmetric matrix and all diagonal entries are zero, then for almost all such H, the interference alignment equations are solvable.

Theorem 4. *For 4-user interference channel with reciprocal feedback channel ($G = H^T$), if H is symmetric channel and all diagonal entries are zero, i.e., H can be written as*

$$H = \begin{pmatrix} 0 & h_1 & h_2 & h_3 \\ h_1 & 0 & h_4 & h_5 \\ h_2 & h_4 & 0 & h_6 \\ h_3 & h_5 & h_6 & 0 \end{pmatrix}, \qquad (53)$$

then for generic $\{h_1, h_2, h_3, h_4, h_5, h_6\}$, i.e., for all $\{h_1, h_2, h_3, h_4, h_5, h_6\}$ except a set of $\{h_1, h_2, h_3, h_4, h_5, h_6\}$ which satisfies a nontrivial polynomial equation, there

exists D_1, D_2 *and* D_3 *such that* $(HD_2 + HD_3GD_1H)$ *is proportional to the corresponding row of* H *except the diagonal entries.*

Proof: We derive the interference alignment equations and then use commercial computer algebra system Maple to compute the corresponding symbolic Gröbner basis, which does not contain a nonzero scalar. Therefore, due to Corollary 4 in Appendix B, we conclude that for generic $\{h_1, h_2, h_3, h_4, h_5, h_6\}$, there exists D_1, D_2 and D_3 such that $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries. ∎

C. $K = 5$ and $K = 6$

Recall that the number of variables N_v is $3K$, the number of equations N_e is $K(K-2)$ and the number of inequalities N_{ie} is K. N_e grows faster than N_v as K increases, and when $K \geq 5$, $N_v \leq N_e$. We expect that when the number of variables is not more than the number of inequations, a valid solution may not exist. Indeed, we confirm this conjecture for $K = 5$ and 6 via computing Gröbner basis for random numeric H and G and finding that 1 is in the Gröbner basis in all instances.

Claim 4. *For K-user interference channel with reciprocal feedback channel ($G = H^T$) where $K = 5$ or 6, for generic matrix H, there does not exist D_1, D_2 and D_3 such that each row of $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries.*

Claim 5. *For K-user interference channel with out-of-band feedback feedback channel where $K = 5$ or 6, for generic matrices H and G, there does not exist D_1, D_2 and D_3 such that each row of $(HD_2 + HD_3GD_1H)$ is proportional to the corresponding row of H except the diagonal entries.*

Since one reverse transmission from destinations to sources is not enough, in Phase 2 (interaction phase) we can let destinations make more than one reverse transmissions to sources. In one extreme case, if each destination sends what it receives in Phase 1 to all sources sequentially, where in total we use K reverse transmissions[3], then each source can compute the symbol signals sent out by all sources in the first phase, and then each source can do precoding over all symbols in the second phase, which reduces to a MIMO broadcast channel.

It turns out that when $K = 5$ or $K = 6$, two reverse transmissions in Phase 2 is sufficient to do interference alignment while still preserving the desired signals. A natural three-phase scheme is as follows:

- Phase 1: All sources send their independent symbols simultaneously. And destinations get $y = Hx + n$, where n is the additive noise of the channel.
- Phase 2-1: After receiving signals from sources in phase1, all destinations scale y and send back to sources using the feedback channel. sources get $f^1 = GD_1y + \text{noise}$.

[3]In fact, it is easy to see that $(K-1)$ reverse transmissions are also sufficient.

- Phase 2-2: All destinations scale y using a different coding matrix and send back to sources. sources get $f^2 = GD_2y + \text{noise}$.
- Phase 3: Now each source has three sets of signals x, f^1 and f^2, and sources can send out a linear combination of x, f^1 and f^2 to destinations via the forward channel. destinations get

$$y' = H(D_3x + D_4f^1 + D_5f^2) + \text{noise} \quad (54)$$
$$= (HD_3 + HD_4GD_1H + HD_5GD_2H)x + \text{noise}. \quad (55)$$

With two reverse transmissions, the number of variables is increased to $5K$ from $3K$, and is more than $N_e = K(K-2)$ for $K = 5, 6$. Extensive numeric simulations show that for generic matrix H and G, there exists diagonal coding matrices $\{D_i\}_{1 \leq i \leq 5}$ to align interferences while still preserving desired signals.

Claim 6. *For K-user interference channel with reciprocal feedback channel ($G = H^T$) where $K = 5$ or 6, for generic matrices H, there exists D_1, D_2, D_3, D_4 and D_5 such that each row of $(HD_3 + HD_4GD_1H + HD_5GD_2H)$ is proportional to the corresponding row of H except the diagonal entries.*

Claim 7. *For K-user interference channel with out-of-band feedback channel where $K = 5$ or 6, for generic matrices H and G, there exists D_1, D_2, D_3, D_4 and D_5 such that each row of $(HD_3 + HD_4GD_1H + HD_5GD_2H)$ is proportional to the corresponding row of H except the diagonal entries.*

VII. INTERACTIVE INTERFERENCE ALIGNMENT FOR SMALL K WITH FULL DUPLEX RADIOS

In this section, we study how to exploit interaction in interference channel where all sources and destinations have full-duplex antennas so that all nodes can transmit and receive in the same band simultaneously. We will show how full-duplex operations can help align interferences and achieve the optimal degrees of freedom for some classes of interference channels.

A. SISO IFC with small K

Recall in the system model of full-duplex IFC, H is the channel matrix from sources to destination, U is the channel matrix among sources, and W is the channel matrix among destinations. Consider the following simple two-phase transmission scheme.

- Phase 1: All sources send out signals x simultaneously. After the transmission, sources get $f = Ux + \text{noise}$, and destinations get $y = Hx + \text{noise}$.
- Phase 2: sources send out a linear combination of x and f, and destinations send out scaled version of y. Therefore, destinations get $H(D_1x + D_2f) + WD_3y + \text{noise} = (HD_1 + HD_2U + WD_3H)x + \text{noise}$, where D_1, D_2 and D_3 are diagonal coding matrices.

Note that in this transmission scheme no feedback channel from destinations to sources is required. Similarly, if each

row of $(HD_1 + HD_2U + WD_3H)$ is proportional to the corresponding row of H except all diagonal entries, then interferences at all destinations are aligned and all destinations can retrieve their designed signals from sources without any interference, and thus achieve the optimal $\frac{K}{2}$ degrees of freedom.

Our first result is that the above two-phase transmission scheme works for $K = 3$ and $K = 4$.

Theorem 5. *For $K = 3$ and $K = 4$, for generic channel matrices U, H and W, there exists diagonal matrices D_1, D_2 and D_3 such that each row of $(HD_1 + HD_2U + WD_3H)$ is proportional to the corresponding row of H except all diagonal entries.*

Proof: The interference alignment equations are linear in D_1, D_2 and D_3, so as in the proof of Theorem 3 we can use dimension argument to show that there always exists a solution to the system of linear interference alignment equations and it also satisfies the inequality constraints. ∎

Using the same dimension argument, we can prove that the above scheme does not work for $K = 5$ or bigger.

B. MIMO IFC with $K = 4$

Next we show that for four-user MIMO interference channel, full-duplex antennas can help design a simple transmission scheme to achieve the optimal $\frac{KM}{2}$ degrees of freedom.

For K-user MIMO interference channel where all sources and destinations are equipped with M antennas, [1] shows that for $K = 3, M > 1$, vector space interference alignment can achieve the optimal $\frac{KM}{2} = \frac{3M}{2}$ degrees of freedom and channel diversity is not required. [3] and [14] prove that in general vector space MIMO interference alignment can at most get $\frac{2KM}{K+1}$ degrees of freedom, which is strictly less than the optimal $\frac{KM}{2}$ when $K \geq 4$. We show that for four-user MIMO interference channel with full-duplex antennas, a simple two-phase transmission scheme can achieve the optimal $2M$ degrees of freedom at least for all $M \leq 15$.

Suppose source s_i wants to send M symbols $x_{(i-1)M+1}, x_{(i-1)M+2}, \ldots, x_{iM}$ to destination t_i, for all $1 \leq i \leq 4$. Let $x \triangleq (x_1, x_2, \ldots, x_{4M})^T$. Consider a natural two-phase transmission scheme for 4-user MIMO IFC with full-duplex antennas:

- Phase 1: All sources send out signals x simultaneously, i.e., the ith antenna sends x_i for all $1 \leq i \leq 4M$. After the transmission, sources get $f = Ux + \text{noise}$, and destinations get $y^1 = Hx + \text{noise}$.
- Phase 2: sources do linear coding over x and f, and send $D_1x + D_2f$, and destinations do linear coding over y^1 and send D_3y, where D_1, D_2 and D_3 are block diagonal matrices with diagonal block size of $M \times M$. Therefore, destinations get

$$
\begin{aligned}
y^2 &= H(D_1x + D_2f) + WD_3y^1 + \text{noise} \\
&= (HD_1 + HD_2U + WD_3H)x + \text{noise}. \quad (56)
\end{aligned}
$$

Note that since each node can only do coding over what signals it has or received from other nodes, the coding matrices

D_1, D_2 and D_3 have to be block diagonal matrices with diagonal block size of $M \times M$.

A sufficient condition for all destinations to retrieve the desired symbols from the corresponding sources without interference is that at each antenna of destination t_i, all interference symbols are aligned except the desired symbols $x_{(i-1)M+1}, x_{(i-1)M+2}, \ldots, x_{iM}$, for all $1 \leq i \leq 4$.

More precisely, let $B \triangleq HD_1 + HD_2U + WD_3H$ and denote the index set $\{(i-1)M+1, (i-1)M+2, \ldots, iM\}$ by I_i. Then for the destination t_i, in Phase 1 it receives

$$
y_m^1 = \sum_{m' \in I_i} H_{mm'}x'_m + \sum_{j \notin I_i} H_{mj}x_j + \text{noise term}, \quad (57)
$$

and in Phase 2 it receives

$$
y_m^2 = \sum_{m' \in I_i} B_{mm'}x'_m + \sum_{j \notin I_i} B_{mj}x_j + \text{noise term}. \quad (58)
$$

The first term on the RHS of (57) and (57) correspond to the desired signals, and the second term are interferences. If at each antenna of t_i, all interferences are aligned, i.e., for all $m \in I_i$, there exists $\lambda_m \in \mathbb{C}$, such that

$$
\frac{B_{mj}}{H_{mj}} = \lambda_m, \quad \forall j \notin I_i, \quad (59)
$$

then by computing $\tilde{y}_m \triangleq y_m^2 - \lambda_m y_m^1$, t_i can cancel all interferences

$$
\begin{aligned}
\tilde{y}_m &\triangleq y_m^2 - \lambda_m y_m^1 \quad (60) \\
&= \sum_{m' \in I_i} (B_{mm'} - \lambda_m B_{mm'})x'_m + \text{noise term}. \quad (61)
\end{aligned}
$$

Now t_i gets k linear equations in k variables $\{x_m\}_{m \in I_i}$. If the coefficient matrix of the k linear equations has full rank, then t_i can retrieve all desired signals $\{x_m\}_{m \in I_i}$ corrupted by noise only.

The condition that the coefficient matrix has full rank is equivalent to the condition that the determinant of the coefficient matrix is nonzero. Therefore, to make the transmission scheme work, we need to find block diagonal coding matrices D_1, D_2, D_3 to satisfy the interference alignment linear equations (59) and make the determinants of coefficient matrices be nonzero. Without introducing the auxiliary variables λ_m, we have $KM((K-1)M-1)$ interference alignment equations, which are linear in terms of D_1, D_2 and D_3. Since the equations are linear, there exists analytical solution to D_1, D_2 and D_3 which are rational functions of U, H and W. Then, we can plug the analytical solution into the expression of the determinant of the coefficient matrix and check whether it is zero. We have checked that for $K = 4$, the determinants of the coefficient matrices are nonzero expressions for all $M \leq 15$.

Theorem 6. *For 4-user MIMO interference channel where each node have M full-duplex antennas, then for generic matrices H, U and W, the two-phase scheme can achieve the optimal $\frac{K}{2}$ degrees of freedom, at least for $M \leq 15$.*
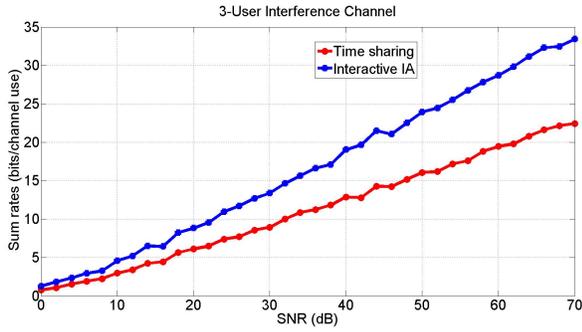
Fig. 6. Numeric Finite SNR Performance Simulations for 3-User Interference Channel with Out-of-Band Reverse (Feedback) Channel.
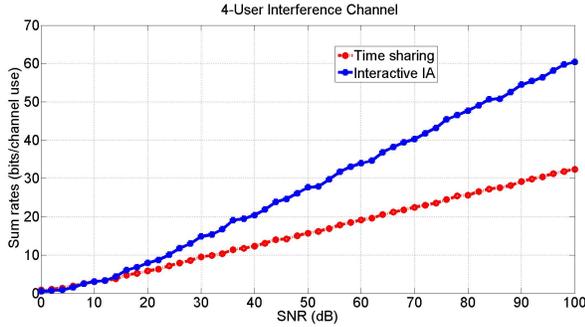

Fig. 7. Numeric Finite SNR Performance Simulations for 4-User Interference Channel with Out-of-Band Reverse (Feedback) Channel.

Note that for two-phase scheme, in total we have $N_v = 3KM^2$ variables and $N_e = KM((K-1)M-1)$ linear equations. When $K = 4$, $N_v = 12M^2 > N_e = 12M^2 - 4M$ for all M, which means we have more variables than equations, and thus we believe Theorem 6 holds for any M.

VIII. NUMERIC FINITE SNR PERFORMANCE EVALUATION

In this section, we present simulation results on the finite SNR performance of our three-phase interactive interference alignment scheme with out-of-band reverse (feedback) transmissions for 3-user and 4-user interference channels, where the channel parameters are drawn from i.i.d. complex Gaussian distribution. We assume both sources and destinations have the same power constraint. We formulate the problem as a non-convex optimization problem, and use generic local algorithm to maximize the sum rates among all possible interference alignment solutions while satisfying power constraints. For each SNR, we simulate over 50 channel realizations and compute the average of maximum sum rates. We compare the maximum rates achieved via interactive interference alignment with the standard time sharing scheme.

For $K = 3$ and $K = 4$, our interactive interference alignment scheme can achieve the optimal $\frac{K}{2}$ degrees of freedom, while time sharing scheme only gets 1 degree of freedom. Therefore, at high SNR, we expect our scheme has 50 percent improvement for $K = 3$ and 100 percent improvement

for $K = 4$ over time sharing scheme, and this matches our simulation results in Figure 6 and Figure 7.

IX. DISCUSSIONS AND OPEN PROBLEMS

In this section, we give a general multi-phase interactive communication scheme for K-user interference channel with large K and discuss how $O(\sqrt{K})$ degrees of freedom might be achieved with reverse transmissions taken into account. We also discuss some open problems following this line of work.

A. Multi-Phase Transmission Scheme for General K

In Section VI, we have shown that for general K we can achieve the optimal $\frac{K}{2}$ degrees of freedom using the three-phase scheme with $(K-1)$ reverse transmissions in Phase 2. However, if we take reverse transmissions into account, the total number of transmissions is $2 + (K-1) = K + 1$, and thus the total degrees of freedom is $\frac{K}{K+1} < 1$. In this section, we give a multi-phase transmission scheme and use equation-variable counting argument to show that we may potentially achieve $O(\sqrt{K})$ degrees of freedom even if we take reverse transmissions into account.

Multi-Phase Transmission Scheme

- Forward Phase 1: All TXs send their independent symbols simultaneously. And RXs get $y^1 = Hx + \text{noise}$.
- Reverse Phase 1: All RXs scale y and send back to TXs using the feedback channel. TXs get $f^1 = GD_1y + \text{noise}$.
- Forward Phase 2: All TXs send a linear combination of x and f^1 to RXs. And RXs get $y^2 = H(D_2x + D_3f^1) + \text{noise}$.
- Reverse Phase 2: All RXs send a linear combination of y^1 and y^2 to RXs. TXs get $f^2 = G(D_4y^1 + D_5y^2) + \text{noise}$.
- $\ldots$
- Forward Phase i: All TXs send a linear combination of $x, f^1, f^2, \ldots, f^{i-1}$ to RXs, using i diagonal coding matrices, and RXs get y^i.
- Reverse Phase i: All RXs send a linear combination of $y^1, y^2, \ldots, y^i$ to RXs, using i diagonal coding matrices, and RXs get y^i.
- $\ldots$
- Forward Phase N: All TXs send a linear combination of $x, f^1, f^2, \ldots, f^{N-1}$ to RXs, using N diagonal coding matrices, and RXs get y^N.

The number of coding matrices is

$$1 + 2 + 2 + 3 + 3 + \cdots + (N-1) + (N-1) + N$$
$$= N^2 - 1, \tag{62}$$

so the total number of variables N_v is $(N^2-1)K$. On the other hand, the total number of interference alignment equations N_e is $K(K-2)$. If we want to have $N_v \geq N_e$, then $N \geq \sqrt{K-1}$, and thus $N = \Omega(\sqrt{K})$. If there exists a solution to the corresponding system of polynomial equations, then the sum degrees of freedom achievable is

$$\frac{K}{2N} \leq \frac{K}{2\sqrt{K-1}} = O(\sqrt{K}). \tag{63}$$

Therefore, by equation-variable counting, we conjecture that the multi-phase interactive interference alignment may achieve $O(\sqrt{K})$ degrees of freedom for general K.

B. Open Problems

We list several interesting open problems following this line of work on interactive interference alignment.

- **Constructive Proof.** While we have used the generic Gröbner basis tool to show whether the interference alignment equations are solvable or not, it will be more interesting to derive an analytical solution, like what we have done in the case where H can be represented as the sum of a diagonal matrix and a rank 1 matrix.
- **Efficient Algorithm and Optimization.** In theory we can use the Buchberger's algorithm to get an interference alignment solution, but the computational complexity is high. So one natural question is whether we can design a much more efficient algorithm to solve the interference alignment equations. In addition, as presented in Section VIII, to maximize the sum rates using interactive interference alignment, we formulate a nonconvex optimization problem and use generic local algorithm to solve it. Can we design an efficient algorithm to solve the optimization problem so that it can be used in practice (in a distributed way)?
- **Half-duplex interactive IA for general K.** We give an equation-variable counting argument to show why $O(\sqrt{K})$ degrees of freedom might be achievable using interactive interactive alignment with reverse transmissions into account. Can one rigorously prove or disprove it? Can we design a new transmissions scheme to get degrees of freedom linearly growing with K using the same idea of interactive interference alignment?
- **Full-duplex interactive IA for general K.** We have shown for small K, e.g., $K = 3$ and 4, interactive IA with full-duplex radios can achieve the optimal degrees of freedom, while for general K the current two phase scheme does not work for $K \geq 5$. It is interesting to study how to extend the two phase interactive IA to general K-user interference channel with full-duplex antennas.
- **Interactive IA with relays.** Relays naturally fit into our model, for both half-duplex and full-duplex radios. Through interaction among relays, sources and destinations, relays can help sources do interference alignment. For instance, for half-duplex K-user interference channel, with aK^2 relays, for some constant a, in the natural three phase scheme with one reverse transmission from the destinations, relays can help align all interferences at each destination. Can one characterize the minimum number of relays needed, or design a new transmission scheme that achieves alignment using $o(K^2)$ relays?

X. Conclusion

We study new channel models where interaction among sources and destinations is enabled, e.g., both sources and destinations can talk to each other. The interaction can come in two ways: 1) for half-duplex radios, destinations can talk back to sources using simultaneous out-of-band transmission or in band half-duplex transmission; 2) for full-duplex radios, both sources and destinations can transmit and listen in the same channel simultaneously. Although [9] shows that for interference channel, relays, feedback, and full-duplex operation cannot improve the degrees of freedom beyond $\frac{K}{2}$, we demonstrate that the interactions among sources and destinations enables flexibility in designing simple interference alignment scheme and in several cases achieves the optimal degrees of freedom.

We present a three-phase interactive communication scheme for half-duplex K-user interference channel with small K, a two-phase interference communication scheme for full-duplex K-user interference channel with small K, and show that the interactive interactive alignment scheme can achieve the optimal degrees of freedom for several classes of IFC, including half-duplex 3-user IFC, full-duplex 3-user and 4-user IFC, and full-duplex 4-user MIMO IFC. We do extensive numeric simulations and show the proposed interactive interactive alignment scheme also works for some other classes of IFC empirically. We use tools from algebraic geometry to show why success of numeric simulations can suggest the scheme should work well for almost all channel parameters in a rigorous way. Simulation results on the finite SNR performance of interactive interactive alignment for 3-user and 4-user half-duplex interference channels are presented. Finally, we discuss a general multi-phase interactive communication scheme for K-user interference channel with large K, and list several open problems.

Appendix A
Proof of Theorem 1

Proof of Theorem 1: Following the notation introduced in Section IV, equivalently, we need to prove that there exists diagonal coding matrices D_1, D_2 and D_3 such that $(HD_2 + HD_3GD_1H)$ is a diagonal matrix and each diagonal entry is strictly nonzero.

We give a constructive proof. First, we expand $(HD_2 + HD_3GD_1H)$ and get

$$HD_2 + HD_3GD_1H \tag{64}$$
$$=(D + uv^T)D_2 + (D + uv^T)D_3(D + vu^T)D_1(D + uv^T) \tag{65}$$
$$=DD_2 + uv^TD_2 + (DD_3 + uv^TD_3)(DD_1 + vu^TD_1)(D + uv^T) \tag{66}$$
$$=DD_2 + uv^TD_2 + (DD_3DD_1 + uv^TD_3DD_1 + uv^TD_3vu^TD_1 + DD_3vu^TD_1)(D + uv^T) \tag{67}$$
$$=DD_2 + uv^TD_2 + DD_3DD_1D + uv^TD_3DD_1D + uv^TD_3vu^TD_1D + DD_3vu^TD_1D + DD_3DD_1uv^T + uv^TD_3DD_1uv^T + uv^TD_3vu^TD_1uv^T + DD_3vu^TD_1uv^T. \tag{68}$$

Define $U \triangleq \operatorname{diag}(u)$, i.e., U is a diagonal matrix where the ith diagonal entry is the ith component of u. Similarly, define $V \triangleq \operatorname{diag}(v)$. Set

$$D_1 = D^{-1}U^{-1}V, \tag{69}$$
$$D_3 = D^{-1}V^{-1}U. \tag{70}$$

Then

$$D_3 D D_1 D = I, \tag{71}$$
$$u^T D_1 D = v^T, \tag{72}$$
$$D D_3 v = u, \tag{73}$$

where I is the identity matrix.

Therefore,

$$HD_2 + HD_3GD_1H \tag{74}$$
$$=DD_2 + uv^T D_2 + D + uv^T + uv^T D_3 vv^T$$
$$+ uv^T + uv^T + uu^T D_1 uv^T +$$
$$uv^T D_3 vu^T D_1 uv^T + uu^T D_1 uv^T \tag{75}$$
$$=D(D_2 + I) + uv^T D_2 + (3 + 2v^T D_3 v + u^T D_1 u + $$
$$v^T D_3 vu^T D_1 u)uv^T \tag{76}$$
$$=D(D_2 + I) + uv^T D_2 +$$
$$(3 + 3v^T D^{-1}u + (v^T D^{-1}u)^2)uv^T. \tag{77}$$

Define $\alpha = 3 + 3v^T D^{-1}u + (v^T D^{-1}u)^2$, and set

$$D_2 = -\alpha I. \tag{78}$$

Then

$$HD_2 + HD_3GD_1H = (1-\alpha)D. \tag{79}$$

Since $\alpha \neq 1$, $(HD_2 + HD_3GD_1H)$ is a diagonal matrix and each diagonal entry is nonzero. ∎

APPENDIX B
MATHEMATICAL BACKGROUND ON ALGEBRAIC GEOMETRY

In the section we present some basic terminologies and results in algebraic geometry. In particular, we introduce the Gröbner basis and Buchberger's algorithm, which are standard tools to check whether a system of polynomial equations have solutions or not. For more details on algebraic geometry, we refer the reader to the excellent textbook [11].

Recall that we use k to denote a field, $k[\xi_1, \xi_2, \ldots, \xi_n]$ to denote the set of all polynomials in $\xi_1, \xi_2, \ldots, \xi_n$ with coefficients in k, and use $V(f_1, \ldots, f_s)$ to denote the affine variety defined by polynomials $f_1, \ldots, f_s \in k[\xi_1, \xi_2, \ldots, \xi_n]$.

Definition 2 (Definition 2 on page 30 of [11]). *Let $f_1, \ldots, f_s$ be polynomials in $k[\xi_1, \xi_2, \ldots, \xi_n]$. Then we denote the ideal generated by $f_1, \ldots, f_s$ as*

$$\langle f_1, \ldots, f_s \rangle \triangleq \{ \sum_{i=1}^{s} h_i f_i \mid h_1, \ldots, h_s \in k[\xi_1, \xi_2, \ldots, \xi_n] \}. \tag{80}$$

It is straightforward to see that

$$V(f_1, \ldots, f_s) = V(\langle f_1, \ldots, f_s \rangle). \tag{81}$$

Theorem 7 (The Weak Nullstellensatz on page 170 of [11]). *Let k be an algebraically closed field and let $I \subset k[\xi_1, \xi_2, \ldots, \xi_n]$ be an ideal satisfying $V(I) = \emptyset$. Then $I = k[\xi_1, \ldots, \xi_n]$.*

Corollary 2. *Let $f_1, \ldots, f_s$ be polynomials in $k[\xi_1, \xi_2, \ldots, \xi_n]$, where k is an algebraically closed field. Then $V(f_1, \ldots, f_s) = \emptyset$ if and only if $1 \in \langle f_1, \ldots, f_s \rangle$.*

Proof: If $1 \in \langle f_1, \ldots, f_s \rangle$, then $\langle f_1, \ldots, f_s \rangle = k[\xi_1, \ldots, \xi_n]$, and thus $V(f_1, \ldots, f_s) = V(\langle f_1, \ldots, f_s \rangle) = V(k[\xi_1, \ldots, \xi_n]) = \emptyset$.

On the other hand, if $V(f_1, \ldots, f_s) = \emptyset$, equivalently, $V(\langle f_1, \ldots, f_s \rangle) = \emptyset$, then by Theorem 7 we have

$$\langle f_1, \ldots, f_s \rangle = k[\xi_1, \ldots, \xi_n], \tag{82}$$

and thus $1 \in \langle f_1, \ldots, f_s \rangle$. ∎

Therefore, given polynomials $f_1, \ldots, f_s \in k[\xi_1, \ldots, \xi_n]$, where k is an algebraically closed field, to check whether $V(f_1, \ldots, f_s)$ is nonempty, i.e., whether there exists solution to the system of polynomial equations

$$f_i = 0, \forall 1 \leq i \leq s, \tag{83}$$

we only need to check whether $1 \in \langle f_1, \ldots, f_s \rangle$. This is the ideal membership problem, which is well studied in algebraic geometry. The basic tool we will use is *Gröbner basis* [12]. Below we give a short introduction to *Gröbner basis* and show how to use *Gröbner basis* to solve the ideal membership problem.

Definition 3 (Definition 7 on page 59 of [11]). *Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[\xi_1, \ldots, \xi_n]$ and let $>$ be a monomial order. Then we define the **multidegree** of f as*

$$multideg(f) \triangleq \max(\alpha : a_\alpha \neq 0), \tag{84}$$

where the maximum is taken with respect to the monomial order $>$.

*And we define the **leading term** of f as*

$$LT(f) \triangleq a_{multideg(f)} \xi^{multideg(f)}. \tag{85}$$

Given an ideal I, we use $LT(I)$ to denote the set of leading terms of elements of I, i.e.,

$$LT(I) \triangleq \{LT(f)|f \in I\}. \tag{86}$$

Due to Theorem 8, every ideal in $k[\xi_1, \ldots, \xi_n]$ has a finite generating set.

Theorem 8 (Hilbert Basis Theorem on page 76 of [11]). *Every ideal $I \subset k[\xi_1, \ldots, \xi_n]$ has a finite generating set. That is, $I = \langle g_1, \ldots, g_t \rangle$ for some $g_1, \ldots, g_t \in I$.*

Definition 4 (Definition 5 on page 77 of [11]). *Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a **Gröbner basis** if*

$$\langle LT(g_1), \ldots, LT(g_t)\rangle = \langle LT(I)\rangle. \tag{87}$$

Equivalently, a set $\{g_1, \ldots, g_t\} \subset I$ is a Gröbner basis of I if and only if the leading term of any element of I is divisible by one of the $LT(g_i)$ [11].

Proposition 1 (Proposition 1 on page 82 of [11]). *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset k[\xi_1, \ldots, \xi_n]$ and let $f \in k[\xi_1, \ldots, \xi_n]$. Then there is a unique $r \in k[\xi_1, \ldots, \xi_n]$ with the following two properties:*
(i) No term of r is divisible by any of $LT(g_1), \ldots, LT(g_t)$.
(ii) There is $g \in I$ such that $f = g + r$.
In particular, r is the remainder on division of f by G no matter how the elements of G are listed when using the division algorithm.

An important property of Gröbner basis is that given a polynomial f, we can check whether f is in the corresponding ideal via division of f by G. More precisely,

Corollary 3 (Corollary 2 on page 82 of [11]). *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset k[\xi_1, \ldots, \xi_n]$ and let $f \in k[\xi_1, \ldots, \xi_n]$. Then $f \in I$ if and only if the remainder on division of f by G is zero.*

Corollary 4. *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset k[\xi_1, \ldots, \xi_n]$. Then $1 \in I$ if and only if*

$$g_i = \lambda, \tag{88}$$

for some $i \leq t$ and nonzero $\lambda \in k$.

    Proof: We first prove the easier direction. If for some $i, g_i = \lambda \neq 0$, then $\langle g_i \rangle = k[\xi_1, \ldots, \xi_n]$. So $I = k[\xi_1, \ldots, \xi_n]$ and thus $1 \in I$.

For the other direction, suppose that $1 \in I$, but for each $i \leq t$, the leading term of g_i has a nonzero order, i.e., g_i does not have the form $g_i = \lambda$ for some nonzero $\lambda \in k$. Then after we do division of 1 by G, the remainder will be 1, so the remainder is nonzero, and thus $1 \notin I$ by Corollary 3, which leads to a contradiction. Therefore, for some i, g_i can be written as $g_i = \lambda$ for some nonzero $\lambda \in k$. ∎

Due to Corollary 2 and Corollary 4, to check whether there exists solution to a system of polynomial equations over algebraically closed field, we only need to compute a Gröbner basis of the corresponding ideal and check whether a nonzero scalar is in the Gröbner basis.

The standard approach to computing a Gröbner basis is Buchberger's algorithm [12], which can be viewed as a generalization of Gaussian elimination for linear systems.

Theorem 9 (Buchberger's Algorithm on page 90 of [11]). *Let $I = \langle f_1, f_2, \ldots, f_s \rangle \neq \{0\}$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:*

Buchberger's algorithm can be proven to terminate in finite number of steps, and one important consequence is that if the

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Algorithm 1 Buchberger's algorithm to compute Gröbner basis

Input: $F = (f_1, f_2, \ldots, f_s)$
Output: a Gröbner basis $G = (g_1, g_2, \ldots, g_t)$ for I, with $F \subset G$

Initialization: $G \leftarrow F$
repeat
    $G' \leftarrow G$
    for each pair $\{p, q\}, p \neq q$ in G' **do**
        $\alpha \leftarrow \text{multideg}(p)$, and $\beta \leftarrow \text{multideg}(q)$
        $\gamma \leftarrow (\gamma_1, \ldots, \gamma_n)$, where $\gamma_i = max(\alpha_i, \beta_i)$ for each i.
        $S(p, q) \leftarrow \frac{\xi^\gamma}{LT(p)} p - \frac{\xi^\gamma}{LT(q)} q$ (note that leading terms will be canceled)
        $S \leftarrow \overline{S(p, q)}^{G'}$, where $\overline{S(p, q)}^{G'}$ the remainder on division of $S(p, q)$ by G' using multivariate division algorithm
        if $S \neq 0$ **then**
            $G \leftarrow G \cup S$
        end if
    end for
until G = G'

---

coefficients in the polynomial equations are rational functions of variables $\{h_i\}$, then except a set of $\{h_i\}$ which satisfies a nontrivial polynomial equations on $\{h_i\}$ and thus has a measure 0, either for all $\{h_i\}$ the polynomial equations have a solution, or for all $\{h_i\}$ the polynomial equations do not have a solution. This result is clearly stated in Theorem 2 in Section V.

APPENDIX C
PROOF OF THEOREM 2

Proof of Theorem 2: Suppose all coefficients in $f_1, \ldots, f_s$ are in symbolic form, and we run Buchberger's algorithm to compute $G = \{g_1, \ldots, g_t\}$, the symbolic Gröbner basis for $\langle f_1, f_2, \ldots, f_s \rangle$. In each step of Buchberger's algorithm, we will get new derived polynomials with symbolic coefficients, which are also rational functions of $h_1, h_2, \ldots, h_m$. Since Buchberger's algorithm is proven to terminate in finite number of steps, the set of all symbolic coefficients, which are all rational functions of $h_1, h_2, \ldots, h_m$, is finite. Let e be the product of all numerators and denominators of the set of symbolic coefficients, and thus e is a nontrivial polynomial of $h_1, h_2, \ldots, h_m$.

For all numeric $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1 \times m}$ such that $e(h_1, h_2, \ldots, h_m) \neq 0$, the structure of numeric Gröbner basis is the same as the structure of symbolic Gröbner basis, and thus either $V(f_1, \ldots, f_s) = \emptyset$ or $V(f_1, \ldots, f_s) \neq \emptyset$, which depends on whether the symbolic Gröbner basis contains a nonzero constant.

Therefore, except the set of $\{(h_1, h_2, \ldots, h_m) \mid e(h_1, h_2, \ldots, h_m) = 0\}$, which has a measure 0, either for all $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1 \times m}$,

$V(f_1, \ldots, f_s) \neq \emptyset$, or for all $(h_1, h_2, \ldots, h_m) \in \mathbb{C}^{1 \times m}$, $V(f_1, \ldots, f_s) = \emptyset$. ∎

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